

Finsler Trudinger-Moser inequalities on $\mathbb{R}^2$ Nguyen Tuan Duy¹ & Le Long Phi^{2,*}¹*Faculty of Economics and Law, University of Finance-Marketing, HCM City 72200, Vietnam;*²*Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam**Email: nguyenduy@ufm.edu.vn, lelongphi@duytan.edu.vn*

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Abstract The first aim of this article is to study the sharp singular (two-weight) Trudinger-Moser inequalities with Finsler norms on $\mathbb{R}^2$. The second goal is to propose a different approach to study a vanishing-concentration-compactness principle for the Trudinger-Moser inequalities and use this to investigate the existence and the nonexistence of the maximizers for the Trudinger-Moser inequalities in the subcritical regions. Finally, by applying our Finsler Trudinger-Moser inequalities to suitable Finsler norms, we derive the sharp affine Trudinger-Moser inequalities which are essentially stronger than the Trudinger-Moser inequalities with standard energy of the gradient.

Keywords Trudinger-Moser inequality, Finsler norm, sharp constants, extremal functions, affine energy

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1 Introduction

Studying sharp versions and extremal functions of geometric and functional inequalities has been the main subject of many mathematicians. This line of research plays important roles in several problems arising in the calculus of variations, partial differential equations, geometry, etc. Among those inequalities, the Sobolev-type inequalities are probably one of the most important and frequently used inequalities in the literature.

To study the prescribed Gauss curvature problem on the two-dimensional sphere $\mathbb{S}^2$, Moser [51] studied an exponential Sobolev-type inequality on $\mathbb{S}^2$ with an optimal constant and a sharp limiting Sobolev inequality, namely the embedding $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$, where $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function

$$\varphi_N(t) = \exp(\alpha |t|^{N/(N-1)}) - 1$$

for some $\alpha > 0$ on any finite-measure domain Ω in the Euclidean space $\mathbb{R}^N$. The latter one has been set up without its best form independently by Yudovich [63], Pohožaev [54] and Trudinger [60]. This embedding can be considered as the sharp border-line Sobolev inequality because it provides information on the optimal target space for the Sobolev embedding in the limiting case. Indeed, it is well known that

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$W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq p^* = \frac{pN}{N-p}$ when $p < N$ and $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q < \infty$ when $p = N$ but $W_0^{1,N}(\Omega) \not\subset L^\infty(\Omega)$.

Using the symmetrization arguments, Moser [51] proved the following theorem.

Theorem A. *Let Ω be a domain with finite measure in the Euclidean N -space $\mathbb{R}^N$ ($N \geq 2$). Then we have*

$$\sup_{u \in W_0^{1,N}(\Omega): \int_\Omega |\nabla u|^N dx \leq 1} \frac{1}{|\Omega|} \int_\Omega [\exp(\alpha_N |u|^{\frac{N}{N-1}}) - 1] dx < \infty \quad (1.1)$$

with the optimal constant $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the area of the surface of the unit N -ball.

The constant $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ is optimal in the sense that if we replace α_N by any number $\alpha > \alpha_N$, then the above supremum is infinite. Actually, it can be checked that the constant α_N is indeed sharp in the following sense:

$$\sup_{u \in W_0^{1,N}(\Omega): \int_\Omega |\nabla u|^N dx \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha_N |u|^{\frac{N}{N-1}}) |u|^a dx = \infty, \quad \forall a > 0$$

by using the Moser-type sequence

$$u_n(x) = \begin{cases} \left(\frac{1}{\omega_{N-1}}\right)^{1/N} \left(\frac{n}{N}\right)^{\frac{N-1}{N}}, & 0 \leq |x| \leq e^{-\frac{n}{N}}, \\ \left(\frac{N}{\omega_{N-1}n}\right)^{1/N} \log\left(\frac{1}{|x|}\right), & e^{-\frac{n}{N}} < |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \quad (1.2)$$

The approach in [51] is the following classical Schwarz rearrangement argument: every function $u \geq 0$ is associated with a radially symmetric function $u^\#$ such that the sublevel-sets of $u^\#$ are balls with the same area as that of the corresponding sublevel-sets of u . Moreover, $u^\#$ is a positive and non-increasing function defined on $B_R(0)$, where $|B_R(0)| = |\Omega|$. Hence, by the layer cake representation, we have

$$\int_\Omega f(u) dx = \int_{B_R(0)} f(u^\#) dx$$

for any function f that is the difference of two monotone functions. In particular, we obtain

$$\|u\|_p = \|u^\#\|_p, \\ \int_\Omega \exp(\alpha |u|^{\frac{n}{n-1}}) dx = \int_{B_R(0)} \exp(\alpha |u^\#|^{\frac{n}{n-1}}) dx.$$

Moreover, the well-known Pólya-Szegő inequality

$$\int_{B_R(0)} |\nabla u^\#|^p dx \leq \int_\Omega |\nabla u|^p dx \quad (1.3)$$

plays a crucial role in the approach of Moser [51].

It can be clearly seen that the Trudinger-Moser inequality (1.1) becomes meaningless on domains of infinite volume. Thus, it becomes interesting and nontrivial to extend such inequalities to infinite-measure domains. In this direction, we state here the following three such results in the Euclidean spaces that could be found in [1, 3, 18, 25, 36, 38, 40, 43] and [49, 55] among others.

Theorem B. *Let $0 \leq \beta < N$ and $0 \leq \alpha < \alpha_N$. It holds that*

$$\text{STM}(\alpha, \beta) := \sup_{u \in W^{1,N}(\mathbb{R}^N): \|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \phi_N \left(\alpha \left(1 - \frac{\beta}{N}\right) |u|^{\frac{N}{N-1}} \right) \frac{dx}{|x|^\beta} < \infty, \quad (1.4)$$

$$\text{TM}(\beta) := \sup_{u \in W^{1,N}(\mathbb{R}^N): \|\nabla u\|_N^N + \|u\|_N^N \leq 1} \int_{\mathbb{R}^N} \phi_N \left(\alpha_N \left(1 - \frac{\beta}{N}\right) |u|^{\frac{N}{N-1}} \right) \frac{dx}{|x|^\beta} < \infty, \quad (1.5)$$

$$\text{TME}(\beta) := \sup_{u \in W^{1,N}(\mathbb{R}^N): \|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \frac{\phi_N(\alpha_N(1 - \frac{\beta}{N})|u|^{\frac{N}{N-1}}) dx}{(1 + |u|^{\frac{N}{N-1}(1 - \frac{\beta}{N})}) |x|^\beta} < \infty. \quad (1.6)$$

Here,

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

Moreover, the constant α_N is sharp.

It was also showed in [9, 38] that (1.4) and (1.5) are indeed equivalent. Moreover, Lam et al. [38] studied the asymptotic behavior of $\text{STM}(\alpha, \beta)$ and the relation between $\text{STM}(\alpha, \beta)$ and $\text{TM}(\beta)$. We note here that the approaches in [1, 3, 25, 43, 49, 55] rely on symmetrization arguments, namely the Schwarz rearrangement, which are not available in many settings. In [22, 34, 35, 38], the authors used rearrangement-free arguments to prove that for $0 \leq \beta < N$ and $a, b > 0$, $\text{TM}_{a,b,\beta}(\alpha) < \infty$ if and only if $\alpha < \alpha_N$ or $b \leq N$ and $\alpha = \alpha_N$. Moreover, the inequality is sharp in the sense that $\text{TM}_{a,b,\beta}(\alpha)$ is infinite if and only if $\alpha > \alpha_N$ or $b > N$ and $\alpha = \alpha_N$. Here,

$$\text{TM}_{a,b,\beta}(\alpha) := \sup_{u \in W^{1,N}(\mathbb{R}^N): \|\nabla u\|_N^a + \|u\|_N^b \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta}.$$

In this sense, we can say that the Trudinger-Moser inequality is subcritical if $\alpha < \alpha_N$ or $b < N$ and $\alpha = \alpha_N$.

By the extremal functions of the Trudinger-Moser type inequalities, the first important result was set up by Carleson and Chang in the celebrated work [8]. More precisely, they proved that the supremum

$$\sup_{u \in W_0^{1,N}(\Omega), \int_{\Omega} |\nabla u|^N dx \leq 1} \frac{1}{|\Omega|} \int_{\Omega} \exp(\alpha_N |u|^{\frac{N}{N-1}}) dx$$

can be achieved when Ω is a Euclidean ball. This result came as a surprise because it has been known that the Sobolev inequality does not have extremal functions supported on any finite ball. Subsequently, the existence of extremal functions has been established in other settings such as arbitrary domains in [12, 15, 23, 44] and Riemannian manifolds in [41, 42, 45].

It is worth mentioning that when there is the presence of weights on both sides of the inequalities, the Schwarz rearrangement is not applicable, even on the Euclidean spaces. Thus, it is interesting to study the Trudinger-Moser inequalities in such cases. In this direction, Dong et al. [20] proved, among other results, that for any $\alpha \leq \alpha_N$ and $0 \leq \beta < N$,

$$\int_{\Omega} \exp\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta} \lesssim \int_{\Omega} \frac{dx}{|x|^\beta} \quad (1.7)$$

for all $u \in W_0^{1,N}(\Omega)$ such that

$$\int_{\Omega} |\nabla u|^N dx \leq 1.$$

Again the constant α_N is sharp. The interesting part is that since the weight $\frac{1}{|x|^\beta}$ appears on both sides of the inequality (1.7), the classical Schwarz rearrangement procedure is not available. Hence, in [20], Dong et al. used a suitable quasiconformal transform to reduce the singular case to the non-weighted case. This extends the results in [2], where Adimurthi and Sandeep used the Schwarz rearrangement to study an interpolation between Moser's inequality and the Hardy inequality in the singular case $0 \leq \beta < N$:

$$\sup_{u \in W_0^{1,N}(\Omega): \int_{\Omega} |\nabla u|^N dx \leq 1} \frac{1}{|\Omega|^{1 - \frac{\beta}{N}}} \int_{\Omega} \exp\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta} \leq c_0 \quad (1.8)$$

for any $\alpha \leq \alpha_N$. Indeed, it is clear that

$$\int_{\Omega} \frac{dx}{|x|^\beta} \leq |\Omega|^{1 - \frac{\beta}{N}}.$$

Hence, (1.7) is an improvement of (1.8). In fact, (1.7) is essentially stronger than (1.8) since the ratio $\frac{\int_{\Omega} \frac{dx}{|x|^{\beta}}}{|\Omega|^{1-\frac{\beta}{N}}}$ is not uniformly bounded from below by any positive constant, as Ω ranges in the set of finite-volume domains.

For the applications of such inequalities to geometric analysis and nonlinear PDEs, please refer to, for example, Chang and Yang [10] and De Figueiredo et al. [16].

The Trudinger-Moser type inequalities with Finsler norms have also been investigated in [61, 64], among others. It is worth mentioning that Finsler geometry, which encompasses Riemannian, Euclidean and Minkowski geometries as special cases, has received much attention over the last two decades because of its functions in various fields of science such as continuum mechanics of solids [14], electromagnetism [7], information geometry [59] and quantum information processing [56]. One of the most prominent examples of Finsler geometry is the anisotropic Euclidean space, i.e., the Euclidean space endowed with a Finsler norm. The importance of such a geometrical context can be found in, for example, the seminal work [62] where Wulff reported classical experiments on crystal growth and stated his principal conclusions including the celebrated Wulff's theorem. Actually, many studies have been devoted to extending classical results to such a geometrical setting. We also mention that many other types of functional inequalities with Finsler norms have been investigated in the literature (see, for example, [4, 17, 50, 57]).

The main goal of this article is to study optimal versions as well as the optimizers for the Trudinger-Moser inequalities on $\mathbb{R}^2$ endowed with a Finsler norm. Our motivation is the possible applications of such inequalities to PDEs involving the Finsler-Laplacian, which is a generalized Laplacian.

Let H be a Finsler norm on $\mathbb{R}^2$ and H^o be its polar function (see Section 2 for definitions and properties). We can assume without loss of generality that the convex closed set $K = \{H(x) \leq 1\}$ has measure $|K| = \omega_2$ and set $\kappa_2 = |\{H^o(x) \leq 1\}|$. Our first main result is the Trudinger-Moser inequality on the finite-measure domain with the Finsler norm in the spirit of Moser [51].

Theorem 1.1. *Suppose $0 \leq \beta < 2$ and $\gamma_2 = 4\kappa_2$. Then there exists a constant $C = C(\beta) > 0$ such that for every $u \in W^{1,2}(\mathbb{R}^2)$ with $0 < |\text{supp}(u)| < \infty$ and $\int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1$,*

$$\int_{\mathbb{R}^2} \frac{\exp(\gamma_2(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^{\beta}} dx \leq C(\beta) \int_{\text{supp}(u)} \frac{1}{H^o(x)^{\beta}} dx.$$

Moreover, the constant γ_2 is sharp.

We note that the nonsingular case $\beta = 0$ was studied in [61, 64] by using the convex symmetrization argument. The attainability of the nonsingular case was also investigated in [65]. However, when $\beta > 0$, the convex rearrangement technique cannot be applied. In this situation, we will use a suitable quasiconformal mapping as in [20, 21, 33, 37, 52] to reduce the problem to the non-weighted case.

Using the same idea, we also study the following two-weight Finsler Trudinger-Moser inequalities on domains of infinite measure.

Theorem 1.2. *Suppose $0 \leq \beta < 2$ and $0 \leq \alpha < \gamma_2$. It holds that*

$$\text{STM}_{\beta}(\alpha) := \sup_{u \in C_0^{\infty}(\mathbb{R}^2): \int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1} \frac{1}{\int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^{\beta}} dx} \int_{\mathbb{R}^2} \frac{\exp(\alpha(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^{\beta}} dx < \infty.$$

γ_2 is sharp. Moreover, $\text{STM}_{\beta}(\alpha)$ can be attained.

Remark 1.3. We will actually prove that when $\alpha \lesssim \gamma_2$,

$$\frac{1}{(\gamma_2 - \alpha)} \lesssim \text{STM}_{\beta}(\alpha) \lesssim \frac{1}{(\gamma_2 - \alpha)}.$$

Here, $a \lesssim b$ means that $a < b$ and $b - a$ is small enough. Also, $a \lesssim b$ means that there is a universal constant $C > 0$ such that $a \leq Cb$.

Remark 1.4. By the Caffarelli-Kohn-Nirenberg inequality [6], we have

$$\int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^{\beta}} dx \lesssim \left(\int_{\mathbb{R}^2} H(\nabla u)^2 dx \right)^{\frac{\beta}{2}} \left(\int_{\mathbb{R}^2} |u(x)|^2 dx \right)^{1-\frac{\beta}{2}}.$$

Hence, we get the following Trudinger-Moser inequality as a consequence of our result Theorem 1.2:

$$\sup_{u \in W^{1,2}(\mathbb{R}^2): \int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1} \frac{1}{\|u\|_2^{2-\beta}} \int_{\mathbb{R}^2} \frac{\exp(\alpha(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx < \infty.$$

Remark 1.5. It can be deduced from the proof that in the case $\beta > 0$, the maximizers for $\text{STM}_\beta(\alpha)$ must be H^o -radially symmetric.

Theorem 1.6. Suppose $0 \leq \beta < 2$, $a > 0$ and $0 < b \leq 2$. Then there exists a constant

$$C = C(a, b, \beta) > 0$$

such that $\forall u \in C_0^\infty(\mathbb{R}^2)$ satisfying $(\int_{\mathbb{R}^2} H(\nabla u)^2 dx)^{\frac{a}{2}} + (\int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^\beta} dx)^{\frac{b}{2}} \leq 1$,

$$\int_{\mathbb{R}^N} \frac{\exp(\gamma_2(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx \leq C(a, b, \beta).$$

Moreover, the constant γ_2 is sharp.

Actually, using the same approach as in [28, 38], we will also prove the following results.

Remark 1.7. Let

$$\text{TM}_{a,b,\beta}(\alpha) = \sup_{(\int_{\mathbb{R}^2} H(\nabla u)^2 dx)^{\frac{a}{2}} + (\int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^\beta} dx)^{\frac{b}{2}} \leq 1} \int_{\mathbb{R}^2} \frac{\exp(\alpha(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx.$$

We will also show the following identity:

$$\sup_{s \in (0, \alpha)} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_\beta(s) = \text{TM}_{a,b,\beta}(\alpha).$$

Remark 1.8. Actually, the condition $b \leq 2$ is optimal in the sense that $\text{TM}_{a,b,\beta}(\gamma_2)$ is infinite if (and only if) $b > 2$.

Remark 1.9. Again, in the case $\beta > 0$, the optimizers for $\text{TM}_{a,b,\beta}(\alpha)$, if they exist, must be H^o -radially symmetric.

The next purpose of this article is to investigate the existence and nonexistence of maximizers for $\text{TM}_{a,b,0}(\cdot)$ in the subcritical case, i.e., $\alpha < \gamma_2$ or $\alpha = \gamma_2$ and $b < 2$. More precisely, using the relation in Remark 1.7 and combining the ideas in [11, 29–32, 39], we would like to show the following theorem.

Theorem 1.10. Suppose $a, b > 0$. There exists $\alpha_{a,b,0} \geq 0$ such that $\text{TM}_{a,b,0}(\alpha)$ cannot be attained for all $0 < \alpha < \alpha_{a,b,0}$ and $\text{TM}_{a,b,0}(\alpha)$ is achieved for all $\alpha_{a,b,0} < \alpha < \gamma_2$. Moreover, $\alpha_{a,b,0} = 0$ when $a > 2$ and $\alpha_{a,b,0} > 0$ when $a \leq 2$. Also, there exists an optimizer for $\text{TM}_{a,b,0}(\gamma_2)$ when $b < 2 < a$ or $b \lesssim 2 \approx a$.

Remark 1.11. Theorem 1.10 is actually still available when $\beta > 0$. However, for the sake of simplicity, we will just consider in this paper the nonsingular case $\beta = 0$.

Finally, we note that our results can be generalized to the N -dimensional case. We choose to work on $\mathbb{R}^2$ just for simplicity.

2 Preliminaries

2.1 The Finsler norm

In this subsection, we briefly recall the definitions of the Finsler norm and its polar function and their properties. A nonnegative function $H \in C^2(\mathbb{R}^N \setminus \{0\})$ is a Finsler norm if it is strictly convex and satisfies the following two features: there exist positive constants $\alpha, \beta > 0$ such that for all $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$,

$$H(\lambda x) = |\lambda|H(x),$$

$$\alpha|x| \leq H(x) \leq \beta|x|.$$

The polar function H^o is defined as follows: for all $x \in \mathbb{R}^N$,

$$H^o(x) = \sup_{y \in \mathbb{R}^N \setminus \{0\}} \frac{x \cdot y}{H(y)}.$$

Actually, H^o is a convex, continuous function on $\mathbb{R}^N$ and for all $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned} H^o(\lambda x) &= |\lambda|H^o(x), \\ \frac{1}{\beta}|x| &\leq H^o(x) \leq \frac{1}{\alpha}|x|. \end{aligned}$$

Moreover, for all $x \in \mathbb{R}^N$,

$$H(x) = (H^o)^o(x) = \sup_{y \in \mathbb{R}^N \setminus \{0\}} \frac{x \cdot y}{H^o(y)}.$$

Proposition 2.1. *Let H be a Finsler norm and H^o be its polar function on $\mathbb{R}^N$. For $x \in \mathbb{R}^N \setminus \{0\}$ and $t \neq 0$, we have*

- (1) $\nabla H(x) \cdot x = H(x)$ and $\nabla H^o(x) \cdot x = H^o(x)$;
- (2) $(\nabla H)(tx) = \frac{t}{|t|}(\nabla H)(x)$ and $(\nabla H^o)(tx) = \frac{t}{|t|}(\nabla H^o)(x)$;
- (3) $H(\nabla H^o(x)) = 1 = H^o(\nabla H(x))$;
- (4) $H^o(x)(\nabla H)(\nabla H^o(x)) = x = H(x)(\nabla H^o)(\nabla H(x))$.

2.2 The convex rearrangement

In this subsection, we will introduce the convex rearrangement studied by Alvino et al. [5]. We can assume without loss of generality that the convex closed set $K = \{H(x) \leq 1\}$ has measure $|K| = \omega_2$. We will say that H is the gauge of K . Of course, H^o is the gauge of $K^o = \{H^o(x) \leq 1\}$. We say that K and K^o are polar to each other. Let $\kappa_2 = |K^o|$.

Denote by μ_u the distribution function of a nonnegative function u , given by

$$\mu_u(t) = |\{x : u(x) > t\}| \quad \text{for } t \geq 0$$

and let u^* be the decreasing rearrangement of u defined by

$$u^*(s) = \sup\{t : \mu_u(t) > s\}, \quad s \in (0, |\text{supp } u|).$$

Then we define the convex rearrangement of u :

$$u^\star(x) = u^*(\kappa_2 |H^o(x)|^2).$$

We note that when H is the regular Euclidean norm $|\cdot|$, then $u^\star = u^\#$, where $u^\#$ is the spherical rearrangement of u . By the properties of the decreasing rearrangement u^* , we have that for every continuous increasing function $\Psi : [0, \infty) \rightarrow [0, \infty)$,

$$\int_{\mathbb{R}^2} \Psi(u) dx = \int_0^\infty \Psi(u^*(t)) dt = \int_{\mathbb{R}^2} \Psi(u^\star) dx.$$

We also have for $u \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} H^p(\nabla u) dx \geq \int_0^\infty (-\mu'_u(t))^{1-p} (2\kappa_2^{1/2} \mu_u(t)^{1-1/2})^p dt.$$

One also has

$$\int_{\mathbb{R}^2} H^p(\nabla u^\star) dx = \int_0^\infty (-\mu'_u(t))^{1-p} (2\kappa_2^{1/2} \mu_u(t)^{1-1/2})^p dt. \quad (2.1)$$

Hence, we get the following Pólya-Szegő principle:

$$\int_{\mathbb{R}^2} H^p(\nabla u) dx \geq \int_{\mathbb{R}^2} H^p(\nabla u^\star) dx, \quad \forall u \in W^{1,p}(\mathbb{R}^2), \quad p \geq 1.$$

Also, it is clear from (2.1) that

$$\left(\int_{\mathbb{R}^2} H^p(\nabla u^\star) dx \right)^{1/p} = \frac{\kappa_2^{1/2}}{\omega_2^{1/2}} \left(\int_{\mathbb{R}^2} |\nabla u^\#|^p dx \right)^{1/p}.$$

We also have the Hardy-Littlewood inequality

$$\int_{\mathbb{R}^2} f(x)g(x) dx \leq \int_{\mathbb{R}^2} f^\star(x)g^\star(x) dx$$

for nonnegative functions f and g vanishing at infinity.

2.3 A useful transform

Let $0 < \beta < 2$. We now define the vector-valued function $L_{2,\beta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L_{2,\beta}(x) = H^o(x)^{\frac{\beta}{2-\beta}} x$. The Jacobian matrix of this function $L_{2,\beta}$ is

$$J_{L_{2,\beta}} = H^o(x)^{\frac{\beta}{2-\beta}} \mathbb{I}_2 + A,$$

where

$$A = \begin{pmatrix} \frac{\beta}{2-\beta} H^o(x)^{\frac{2\beta-2}{2-\beta}} x_1 x_1^* & \frac{\beta}{2-\beta} H^o(x)^{\frac{2\beta-2}{2-\beta}} x_1 x_2^* \\ \frac{\beta}{2-\beta} H^o(x)^{\frac{2\beta-2}{2-\beta}} x_2 x_1^* & \frac{\beta}{2-\beta} H^o(x)^{\frac{2\beta-2}{2-\beta}} x_2 x_2^* \end{pmatrix}.$$

Here, we let $x^* = \nabla H^o(x)$. Then

$$\det(\lambda \mathbb{I}_2 - A) = \lambda^2 - \frac{\beta}{2-\beta} H^o(x)^{\frac{\beta}{2-\beta}} \lambda.$$

Choosing $\lambda = -H^o(x)^{\frac{\beta}{2-\beta}}$, we get

$$\det(J_{L_{2,\beta}}) = \det(-H^o(x)^{\frac{\beta}{2-\beta}} \mathbb{I}_2 - A) = \frac{2}{2-\beta} H^o(x)^{2\frac{\beta}{2-\beta}}.$$

Hence, we have

$$\det(J_{L_{2,\beta}}) = \frac{2}{2-\beta} H^o(x)^{2\frac{\beta}{2-\beta}}. \quad (2.2)$$

We now define mappings $D_{2,\beta}$ by

$$D_{2,\beta} u(x) := \left(\frac{2-\beta}{2} \right)^{\frac{1}{2}} u(L_{2,\beta}(x)) = \left(\frac{2-\beta}{2} \right)^{\frac{1}{2}} u(H^o(x)^{\frac{\beta}{2-\beta}} x). \quad (2.3)$$

We also define $D_{2,\beta}^{-1}$ by

$$D_{2,\beta}^{-1} u = v \quad \text{if } u = D_{2,\beta} v.$$

Actually,

$$D_{2,\beta}^{-1} u(x) = \left(\frac{2}{2-\beta} \right)^{\frac{1}{2}} u(H^o(x)^{-\frac{\beta}{2}} x).$$

Then, we have the following results whose proofs can be found in [37].

Lemma 2.2. *The equality in the Cauchy-Schwarz inequality occurs, i.e., for a.e. $x \in \mathbb{R}^2$,*

$$|x \cdot \nabla u(x)| = H^o(x) H(\nabla u(x))$$

if and only if u is H^o -radial, i.e., $u(x) = u(y)$ if $H^o(x) = H^o(y)$.

Lemma 2.3. (1) For a measurable function f , we have

$$\int_{\mathbb{R}^2} \frac{f((\frac{2-\beta}{2})^{\frac{1}{2}}u(x))}{H^o(x)^\beta} dx = \frac{2}{2-\beta} \int_{\mathbb{R}^2} f(D_{2,\beta}u(x)) dx.$$

In particular, we obtain $u \in L^2(\frac{dx}{H^o(x)^\beta})$ if and only if $D_{2,\beta}u \in L^2(dx)$.

(2) If $\nabla u \in L^p(dx)$, then $\nabla D_{2,\beta}u \in L^p(dx)$. Moreover,

$$\int_{\mathbb{R}^2} H(\nabla(D_{2,\beta}u)(x))^p dx \leq \int_{\mathbb{R}^2} H(\nabla u(x))^p dx.$$

The equality occurs if and only if u is H^o -radial.

2.4 The Moser-type sequences

Motivated by the Moser sequence in [51], we will consider the following Finsler-Moser sequences:

$$u_n(x) = \begin{cases} 0, & H^o(x) \geq 1, \\ \left(\frac{1}{\kappa_2 n}\right)^{1/2} \log\left(\frac{1}{H^o(x)}\right), & e^{-\frac{n}{2}} < H^o(x) < 1, \\ \left(\frac{1}{2\kappa_2}\right)^{\frac{1}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}}, & 0 \leq H^o(x) \leq e^{-\frac{n}{2}} \end{cases} \quad (2.4)$$

in order to verify the sharpness of the constants in the Finsler Trudinger-Moser type inequalities.

We have

$$H(\nabla u_n(x)) = |v'_n(H^o(x))|,$$

where

$$v_n(r) = \begin{cases} 0, & r \geq 1, \\ \left(\frac{1}{\kappa_2 n}\right)^{1/2} \log\left(\frac{1}{r}\right), & e^{-\frac{n}{2}} < r < 1, \\ \left(\frac{1}{2\kappa_2}\right)^{\frac{1}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}}, & 0 \leq r \leq e^{-\frac{n}{2}}. \end{cases} \quad (2.5)$$

Hence,

$$\int_{\mathbb{R}^2} H(\nabla u_n)^2 dx = 2\kappa_2 \int_{e^{-\frac{n}{2}}}^1 \frac{1}{\kappa_2 n} r^{-2} r dr = 1.$$

Also,

$$\int_{\mathbb{R}^2} |u_n|^2 dx = 2\kappa_2 \left[\int_0^{e^{-\frac{n}{2}}} \left[\left(\frac{1}{2\kappa_2}\right)^{\frac{1}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}} \right]^2 r dr + \int_{e^{-\frac{n}{2}}}^1 \left[\left(\frac{1}{\kappa_2 n}\right)^{1/2} \log\left(\frac{1}{r}\right) \right]^2 r dr \right] = O\left(\frac{1}{n}\right).$$

3 Trudinger-Moser inequalities on domains of finite measure with arbitrary norms—the proof of Theorem 1.1

Case 1. $\beta = 0$. By $\int_{\mathbb{R}^2} H(\nabla u^\star)^2 dx \leq \int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1$, we first have

$$\begin{aligned} \int_{\mathbb{R}^2} [\exp(\gamma_2 |u(x)|^2) - 1] dx &= \int_{\mathbb{R}^2} [\exp(\gamma_2 |u^\star(x)|^2) - 1] dx \\ &\leq \int_{\mathbb{R}^2} \left[\exp\left(4\pi \frac{|u^\#(x)|^2}{\int_{\mathbb{R}^2} |\nabla u^\#|^2 dx}\right) - 1 \right] dx \\ &\leq C |\text{supp}(u)|. \end{aligned}$$

Here, we used the fact that

$$|\operatorname{supp} u| = |\operatorname{supp} u^\star| = |\operatorname{supp} u^\#|.$$

Case 2. $0 < \beta < 2$. We first note that in this case, since the weight appears on two sides of the inequality, the theory of the convex rearrangement in Subsection 2.1 is not available here. In this case, we will perform the transform introduced in Subsection 2.2. For any $u \in W^{1,2}(\mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1$, we define $v = D_{2,\beta} u$. Then, by Lemma 2.3, we get

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\exp(\gamma_2(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx &= \frac{2}{2 - \beta} \int_{\mathbb{R}^2} [\exp(\gamma_2|v(x)|^2) - 1] dx, \\ \int_{\operatorname{supp}(u)} \frac{1}{H^o(x)^\beta} dx &= \frac{2}{2 - \beta} \int_{\operatorname{supp}(v)} 1 dx \end{aligned}$$

and

$$\int_{\mathbb{R}^2} H(\nabla v)^2 dx \leq \int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\exp(\gamma_2(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx &= \frac{2}{2 - \beta} \int_{\mathbb{R}^2} [\exp(\gamma_2|v(x)|^2) - 1] dx \\ &\leq C(\beta) \int_{\operatorname{supp}(v)} 1 dx \leq C(\beta) \int_{\operatorname{supp}(u)} \frac{1}{H^o(x)^\beta} dx. \end{aligned}$$

Since $\alpha_2 = 4\omega_2$ is sharp when H^o is the regular Euclidean norm $|\cdot|$, it is easy to see that γ_2 is sharp. Another way to see that γ_2 is the best constant is to use the Moser-type sequence (u_n) and

$$w_n(x) = D_{2,\beta}^{-1} u_n(x).$$

Indeed, by $\beta = 0$ and $\alpha > \gamma_2$,

$$\begin{aligned} \int_{\mathbb{R}^2} [\exp(\alpha|u_n(x)|^2) - 1] dx &\geq 2\kappa_2 \int_0^{e^{-\frac{n}{2}}} \left[\exp\left(\alpha \left|\left(\frac{1}{2\kappa_2}\right)^{\frac{1}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}}\right|^2\right) - 1 \right] r dr \\ &\gtrsim \left[\exp\left(\frac{\alpha}{\gamma_2} n\right) - 1 \right] e^n \rightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Also, for $\beta > 0$, by $w_n(x) = D_{2,\beta}^{-1} u_n(x)$, we also have

$$\begin{aligned} \int_{\mathbb{R}^2} H(\nabla w_n)^2 dx &= \int_{\mathbb{R}^2} H(\nabla u_n)^2 dx = 1, \\ \int_{\operatorname{supp}(w_n)} \frac{1}{H^o(x)^\beta} dx &\lesssim \int_{\operatorname{supp}(u_n)} 1 dx \end{aligned}$$

and

$$\int_{\mathbb{R}^2} \frac{\exp(\alpha(1 - \frac{\beta}{2})|w_n(x)|^2) - 1}{H^o(x)^\beta} dx \gtrsim \int_{\mathbb{R}^2} [\exp(\alpha|u_n(x)|^2) - 1] dx \rightarrow \infty.$$

4 Trudinger-Moser inequalities on domains of infinite measure—proofs of Theorems 1.2 and 1.6

4.1 The proof of Theorem 1.2

Proof of Theorem 1.2 for $\beta = 0$. It is enough to consider $\alpha \lesssim \gamma_2$. Let $u \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$, $u \geq 0$ and

$$\int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1.$$

Without loss of generality, we can assume that $\int_{\mathbb{R}^2} |u(x)|^2 dx = 1$. Indeed, set

$$w(x) = u(\lambda x), \quad \lambda = \left(\int_{\mathbb{R}^2} |u(x)|^2 dx \right)^{1/2} > 0.$$

Then it is easy to check that

$$\begin{aligned} \int_{\mathbb{R}^2} |w(x)|^2 dx &= \int_{\mathbb{R}^2} |u(\lambda x)|^2 dx = \frac{1}{\lambda^2} \int_{\mathbb{R}^2} |u(\lambda x)|^2 d(\lambda x) = 1, \\ \frac{1}{\int_{\mathbb{R}^2} |u(x)|^2 dx} \int_{\mathbb{R}^2} [\exp(\alpha |u(x)|^2) - 1] dx &= \int_{\mathbb{R}^2} [\exp(\alpha |w(x)|^2) - 1] dx \end{aligned}$$

and

$$\int_{\mathbb{R}^2} \|\nabla w\|_*^2 dx = \int_{\mathbb{R}^2} \|\lambda(\nabla u)(\lambda x)\|_*^2 dx = \int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1.$$

Set

$$\Omega = \left\{ u(x) > \left(1 - \frac{\alpha}{\gamma_2} \right)^{\frac{1}{2}} \right\}$$

and

$$v(x) = \begin{cases} u(x) - \left(1 - \frac{\alpha}{\gamma_2} \right)^{\frac{1}{2}}, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $v \in W^{1,2}(\mathbb{R}^2)$ has compact support and $|\text{supp}(v)| \lesssim \frac{1}{\gamma_2 - \alpha}$. Moreover,

$$\int_{\mathbb{R}^2} H(\nabla v)^2 dx \leq 1.$$

By $\varepsilon = \frac{\gamma_2 - \alpha}{2\alpha} > 0$, we also have

$$\begin{aligned} |u(x)|^2 &\leq \left(v(x) + \left(1 - \frac{\alpha}{\gamma_2} \right)^{\frac{1}{2}} \right)^2 \\ &\leq (1 + \varepsilon) |v(x)|^2 + \left(1 - \frac{1}{1 + \varepsilon} \right)^{-1} \left| \left(1 - \frac{\alpha}{\gamma_2} \right)^{\frac{1}{2}} \right|^2 \\ &\leq \frac{\gamma_2 + \alpha}{2\alpha} |v(x)|^2 + C. \end{aligned}$$

Hence, we have

$$\begin{aligned} \int_{\mathbb{R}^2} [\exp(\alpha |u(x)|^2) - 1] dx &\lesssim \int_{\mathbb{R}^2} |u(x)|^2 dx + \int_{\Omega} \exp(\alpha |u(x)|^2) dx \\ &\lesssim 1 + \int_{\Omega} \exp \left(\alpha \left(\frac{\gamma_2 + \alpha}{2\alpha} |v(x)|^2 + C \right) \right) dx \\ &\lesssim 1 + |\text{supp}(v)| + |\Omega| \\ &\lesssim \frac{1}{(\gamma_2 - \alpha)}. \end{aligned}$$

When $\alpha = \gamma_2$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} [\exp(\gamma_2 |u_n(x)|^2) - 1] dx &\gtrsim \int_0^{e^{-\frac{n}{2}}} \exp \left(\gamma_2 \left| \left(\frac{1}{2\kappa_2} \right)^{\frac{1}{2}} \left(\frac{n}{2} \right)^{\frac{1}{2}} \right|^2 \right) r dr \\ &\gtrsim e^{-n} e^n. \end{aligned}$$

Hence,

$$\frac{1}{\|u_n\|_2^2} \int_{\mathbb{R}^2} \exp(\gamma_2 |u_n(x)|^2) - 1 \gtrsim \frac{e^{-n} e^n}{O(\frac{1}{n})} \rightarrow \infty.$$

Now, by $\alpha \lesssim \gamma_2$,

$$\int_{\mathbb{R}^2} [\exp(\alpha|u_n|^2) - 1] dx \gtrsim \int_0^{e^{-\frac{\alpha}{\gamma_2}}} \left[\exp\left(\frac{\alpha}{\gamma_2} n\right) - 1 \right] r dr \gtrsim \left[\exp\left(\frac{\alpha}{\gamma_2} n\right) - 1 \right] e^{-n}.$$

We note that there exists a large constant M_2 independent of α such that for $n \geq M_2$,

$$\exp\left(\frac{\alpha}{\gamma_2} n\right) - 1 \gtrsim e^{(\frac{\alpha}{\gamma_2})n}$$

as long as $\frac{\alpha}{\gamma_2} \geq \frac{1}{2}$.

Hence,

$$\int_{\mathbb{R}^2} [\exp(\alpha|u_n|^2) - 1] dx \gtrsim e^{-(1-\frac{\alpha}{\gamma_2})n}.$$

Also for $\alpha \lesssim \gamma_2$, we can pick n such that $1 \leq (1 - \frac{\alpha}{\gamma_2})n \leq 2$, i.e., $\alpha \approx C(1 - \frac{1}{n})\gamma_2$ or $n \approx C\frac{1}{1-\frac{\alpha}{\gamma_2}}$. Then

$$\frac{1}{\|u_n\|_2^2} \int_{\mathbb{R}^2} [\exp(\alpha|u_n|^2) - 1] dx \gtrsim ne^{-2} \approx C\left(\frac{1}{1-\frac{\alpha}{\gamma_2}}\right), \quad (4.1)$$

which implies

$$\text{STM}_0(\alpha) \gtrsim \frac{1}{\gamma_2 - \alpha}.$$

The attainability of $\text{STM}_0(\alpha)$ can be showed by using the same approaches as in [9, 27]. $\square$

Proof of Theorem 1.2 for $\beta > 0$. We perform the change of variables and set $v = D_{2,\beta}u$ with $d = \frac{2}{2-\beta}$. Then, by Lemma 2.3, we get

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\exp(\alpha(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx &= \frac{2}{2-\beta} \int_{\mathbb{R}^2} [\exp(\alpha|v(x)|^2) - 1] dx, \\ \int_{\mathbb{R}^2} \frac{|u|^2}{H^o(x)^\beta} dx &= \left(\frac{2}{2-\beta}\right)^2 \int_{\mathbb{R}^2} |v|^2 dx \end{aligned}$$

and

$$\int_{\mathbb{R}^2} H(\nabla v)^2 dx \leq \int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\exp(\alpha(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx &\lesssim \int_{\mathbb{R}^2} [\exp(\alpha|v(x)|^2) - 1] dx \\ &\lesssim \frac{1}{(\gamma_2 - \alpha)} \int_{\mathbb{R}^2} |v|^2 dx \\ &\lesssim \frac{1}{(\gamma_2 - \alpha)} \int_{\mathbb{R}^2} \frac{|u|^2}{H^o(x)^\beta} dx. \end{aligned}$$

It is easy to check that the sequence $w_n(x) = D_{2,\beta}^{-1}u_n(x)$, where (u_n) is the Moser-type sequence, verifies that γ_2 is sharp. As above, we also have

$$\text{STM}_\beta(\alpha) \gtrsim \frac{1}{(\gamma_N - \alpha)}.$$

By the same argument, we see that if v is an H^o -radially symmetric maximizer for $\text{STM}_0(\alpha)$ (by the above proof and the rearrangement arguments), then $u = D_{2,\beta}^{-1}v$ is an optimizer for $\text{STM}_\beta(\alpha)$. $\square$

4.2 The proof of Theorem 1.6

Proof of Theorem 1.6 for $\beta = 0$. For any $u \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$ satisfying

$$\left(\int_{\mathbb{R}^2} H(\nabla u)^2 dx \right)^{\frac{a}{2}} + \left(\int_{\mathbb{R}^2} |u(x)|^2 dx \right)^{\frac{b}{2}} \leq 1,$$

we set

$$\theta = \left(\int_{\mathbb{R}^2} H(\nabla u)^2 dx \right)^{\frac{1}{2}} \in (0, 1).$$

Hence, $\|u\|_2^b \leq 1 - \theta^a$. If $\frac{1}{2} \leq \theta < 1$, then by defining a new smooth function

$$v(x) = \frac{u(\lambda x)}{\theta}, \quad \text{where } \lambda = \frac{(1 - \theta^a)^{\frac{1}{b}}}{\theta},$$

we can check that

$$\begin{aligned} \int_{\mathbb{R}^2} H(\nabla v)^2 dx &= \frac{1}{\theta^2} \int_{\mathbb{R}^2} H(\nabla u)^2 dx = 1, \\ \int_{\mathbb{R}^2} |v(x)|^2 dx &= \frac{1}{\theta^2} \frac{\|u\|_2^2}{\lambda^2} \leq \frac{(1 - \theta^a)^{\frac{2}{b}}}{\theta^2 \lambda^2} = 1. \end{aligned}$$

Hence, using Theorem 1.2, we get

$$\begin{aligned} \int_{\mathbb{R}^2} [\exp(\gamma_2 |u(x)|^2) - 1] dx &\leq \lambda^2 \int_{\mathbb{R}^2} [\exp(\gamma_2 |\theta v(x)|^2) - 1] dx \\ &\leq \frac{(1 - \theta^a)^{\frac{2}{b}}}{\theta^2} \int_{\mathbb{R}^2} [\exp(\theta^2 \gamma_2 |v(x)|^2) - 1] dx \\ &\lesssim \frac{(1 - \theta^a)^{\frac{2}{b}}}{\theta^2} \frac{1}{(\gamma_2 - \theta^2 \gamma_2)} \\ &\lesssim 1, \quad \text{since } b \leq 2. \end{aligned}$$

If $0 < \theta < \frac{1}{2}$, then by Theorem 1.2,

$$\begin{aligned} \int_{\mathbb{R}^2} [\exp(\gamma_2 |u(x)|^2) - 1] dx &\leq \int_{\mathbb{R}^2} \left[\exp \left(\theta^2 \gamma_2 \left| \frac{u(x)}{\theta} \right|^2 \right) - 1 \right] dx \\ &\leq \int_{\mathbb{R}^2} \left[\exp \left(\frac{\gamma_2}{2^2} \left| \frac{u(x)}{\theta} \right|^2 \right) - 1 \right] dx \lesssim 1. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.6 for $0 < \beta < 2$. Again, we perform the change of variables introduced in Subsection 2.2, and set $v = D_{2,\beta} u$ with $d = \frac{2}{2-\beta}$. Then, by Lemma 2.3, we get

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\exp(\gamma_2 (1 - \frac{\beta}{2}) |u(x)|^2) - 1}{H^o(x)^\beta} dx &= \frac{2}{2 - \beta} \int_{\mathbb{R}^2} [\exp(\gamma_2 |v(x)|^2) - 1] dx, \\ \int_{\mathbb{R}^2} \frac{|u|^2}{H^o(x)^\beta} dx &= \left(\frac{2}{2 - \beta} \right)^2 \int_{\mathbb{R}^2} |v|^2 dx \end{aligned}$$

and

$$\int_{\mathbb{R}^2} H(\nabla v)^2 dx \leq \int_{\mathbb{R}^2} H(\nabla u)^2 dx.$$

Hence,

$$\int_{\mathbb{R}^2} H(\nabla v)^2 dx + \left(\frac{2}{2 - \beta} \right)^2 \int_{\mathbb{R}^2} |v|^2 dx \leq \int_{\mathbb{R}^2} H(\nabla u)^2 dx + \int_{\mathbb{R}^2} \frac{|u|^2}{H^o(x)^\beta} dx \leq 1$$

and

$$\int_{\mathbb{R}^2} \frac{\exp(\gamma_2(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx \lesssim \int_{\mathbb{R}^2} [\exp(\gamma_2|v(x)|^2) - 1] dx \lesssim 1.$$

Finally, using the Moser-type sequence, we can verify that γ_2 is sharp. $\square$

Proof of Remark 1.7. For any $0 < \alpha < \gamma_2$ and $u \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$ satisfying $\int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1$, we set

$$v(x) = \left(\frac{\alpha}{\gamma_2}\right)^{\frac{1}{2}} u(\lambda x).$$

Then

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|v(x)|^2}{H^o(x)^\beta} dx &= \frac{\alpha}{\gamma_2} \int_{\mathbb{R}^2} \frac{|u(\lambda x)|^2}{H^o(x)^\beta} dx = \frac{\alpha}{\gamma_2} \frac{1}{\lambda^{2-\beta}} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^\beta} dx, \\ \int_{\mathbb{R}^2} H(\nabla v)^2 dx &= \frac{\alpha}{\gamma_2} \int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq \frac{\alpha}{\gamma_2}. \end{aligned}$$

Hence, if we choose $\lambda > 0$ such that

$$\left[\frac{\alpha}{\gamma_2} \frac{1}{\lambda^{2-\beta}} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^\beta} dx \right]^{\frac{b}{2}} + \left(\frac{\alpha}{\gamma_2} \right)^{\frac{a}{2}} = 1,$$

i.e.,

$$\lambda = \left[\frac{\frac{\alpha}{\gamma_2} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^\beta} dx}{\left[1 - \left(\frac{\alpha}{\gamma_2}\right)^{\frac{a}{2}}\right]^{\frac{2}{b}}} \right]^{\frac{1}{2-\beta}} > 0,$$

then

$$\int_{\mathbb{R}^2} \frac{|v(x)|^2}{H^o(x)^\beta} dx = \frac{\alpha}{\gamma_2} \frac{1}{\lambda^{2-\beta}} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^\beta} dx = \left[1 - \left(\frac{\alpha}{\gamma_2}\right)^{\frac{a}{2}}\right]^{\frac{2}{b}}$$

and

$$\left(\int_{\mathbb{R}^2} H(\nabla v)^2 dx \right)^{\frac{a}{2}} + \left(\int_{\mathbb{R}^2} \frac{|v(x)|^2}{H^o(x)^\beta} dx \right)^{\frac{b}{2}} \leq 1.$$

Hence,

$$\begin{aligned} & \frac{1}{\int_{\mathbb{R}^2} \frac{|u(x)|^2}{H^o(x)^\beta} dx} \int_{\mathbb{R}^2} \frac{\exp(\alpha(1 - \frac{\beta}{2})|u(x)|^2) - 1}{H^o(x)^\beta} dx \\ &= \frac{\alpha}{\gamma_2} \frac{1}{\int_{\mathbb{R}^2} \frac{|v(x)|^2}{H^o(x)^\beta} dx} \int_{\mathbb{R}^2} \frac{\exp(\gamma_2(1 - \frac{\beta}{2})|v(x)|^2) - 1}{H^o(x)^\beta} dx \\ &= \frac{\frac{\alpha}{\gamma_2}}{\left[1 - \left(\frac{\alpha}{\gamma_2}\right)^{\frac{a}{2}}\right]^{\frac{2}{b}}} \int_{\mathbb{R}^2} \frac{\exp(\gamma_2(1 - \frac{\beta}{2})|v(x)|^2) - 1}{H^o(x)^\beta} dx \\ &\leq \frac{\frac{\alpha}{\gamma_2}}{\left[1 - \left(\frac{\alpha}{\gamma_2}\right)^{\frac{a}{2}}\right]^{\frac{2}{b}}} \text{TM}_{a,b,\beta}(\gamma_2). \end{aligned}$$

Hence,

$$\sup_{\alpha \in (0, \gamma_2)} \frac{\left[1 - \left(\frac{\alpha}{\gamma_2}\right)^{\frac{a}{2}}\right]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_\beta(\alpha) \leq \text{TM}_{a,b,\beta}(\gamma_2).$$

Noting that the above process can be reversed, we can conclude that

$$\sup_{\alpha \in (0, \gamma_2)} \frac{\left[1 - \left(\frac{\alpha}{\gamma_2}\right)^{\frac{a}{2}}\right]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_\beta(\alpha) = \text{TM}_{a,b,\beta}(\gamma_2).$$

Similarly, we could show that

$$\sup_{s \in (0, \alpha)} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_{\beta}(s) = \text{TM}_{a,b,\beta}(\alpha).$$

This completes the proof. $\square$

Proof of Remark 1.8. Assume that for some $b > 2$, we have $\text{TM}_{a,b,\beta}(\gamma_2) < \infty$. Then, similar to the above, we get

$$\overline{\lim}_{\alpha \uparrow \gamma_2} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_{\beta}(\alpha) < \infty.$$

Also, by Theorem 1.2,

$$\underline{\lim}_{\alpha \uparrow \gamma_2} \frac{[1 - \frac{\alpha}{\gamma_2}]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_{\beta}(\alpha) > 0.$$

Hence,

$$\underline{\lim}_{\alpha \uparrow \gamma_2} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]^{\frac{2}{b}}}{1 - \frac{\alpha}{\gamma_2}} < \infty,$$

which is impossible if $b > 2$. $\square$

5 The proof of Theorem 1.10

Lemma 5.1. $\text{STM}_0(\cdot)$ is continuous on $(0, \gamma_2)$. Also, $\text{STM}_0(s) \downarrow 0$ as $s \downarrow 0$ and $\text{STM}_0(s) \uparrow \infty$ as $s \uparrow \gamma_2$.

Proof. From Remark 1.7, we can conclude that $\text{STM}_0(s) \downarrow 0$ as $s \downarrow 0$ and $\text{STM}_0(s) \uparrow \infty$ as $s \uparrow \gamma_2$. Now, fix $\alpha \in (0, \gamma_2)$ and let $\varepsilon_n \downarrow 0$. By Theorem 1.2, rearrangement arguments and the scaling invariant form of STM_0 , we could find an H^o -radially symmetric non-increasing function u_n such that

$$\int_{\mathbb{R}^2} H(\nabla u_n)^2 dx \leq 1, \quad \int_{\mathbb{R}^2} |u_n(x)|^2 dx = 1$$

and

$$\text{STM}_0(\alpha + \varepsilon_n) = \int_{\mathbb{R}^2} [\exp((\alpha + \varepsilon_n)|u_n(x)|^2) - 1] dx > 0.$$

Then

$$\begin{aligned} 0 &\leq \text{STM}_0(\alpha + \varepsilon_n) - \text{STM}_0(\alpha) \\ &\leq \int_{\mathbb{R}^2} [\exp((\alpha + \varepsilon_n)|u_n(x)|^2) - \exp(\alpha|u_n(x)|^2)] dx \\ &= \left(\int_{H^o(x) \geq R} + \int_{H^o(x) < R} \right) \exp(\alpha|u_n(x)|^2) [\exp(\varepsilon_n|u_n(x)|^2) - 1] dx, \end{aligned}$$

where we will choose $R > 0$ such that

$$R^2 \int_{H^o(x) < 1} 1 dx \geq 1.$$

Then

$$1 = \int_{\mathbb{R}^2} |u_n(x)|^2 dx \geq \int_{H^o(x) < R} |u_n(x)|^2 dx \geq |u_n(R)|^2 \int_{H^o(x) < R} 1 dx = |u_n(R)|^2 R^2 \int_{H^o(x) < 1} 1 dx$$

and thus,

$$|u_n(x)| \leq 1 \quad \text{for all } H^o(x) \geq R.$$

Then

$$\begin{aligned} \int_{H^o(x) \geq R} \exp(\alpha |u_n(x)|^2) [\exp(\varepsilon_n |u_n(x)|^2) - 1] dx &\leq e^\alpha \int_{H^o(x) \geq R} \sum_{j=1}^{\infty} \frac{(\varepsilon_n |u_n(x)|^2)^j}{j!} dx \\ &\leq e^\alpha \int_{H^o(x) \geq R} \varepsilon_n e^{\varepsilon_n} |u_n(x)|^2 dx \downarrow 0. \end{aligned}$$

Also, for some $q \gtrsim 1$ we have

$$\begin{aligned} &\int_{H^o(x) < R} \exp(\alpha |u_n(x)|^2) [\exp(\varepsilon_n |u_n(x)|^2) - 1] dx \\ &\leq \left[\int_{H^o(x) < R} [\exp(\alpha q |u_n(x)|^2) - 1 + 1] dx \right]^{1/q} \left[\int_{H^o(x) < R} [\exp(q' \varepsilon_n |u_n(x)|^2) - 1] dx \right]^{1/q'} \\ &\leq [\text{STM}_0(\alpha q) + C(R)]^{1/q} [\text{STM}_0(q' \varepsilon_n)]^{1/q'} \downarrow 0. \end{aligned}$$

Hence, $\text{STM}_0(\alpha + \varepsilon_n) - \text{STM}_0(\alpha) \downarrow 0$. Similarly, $\text{STM}_0(\alpha) - \text{STM}_0(\alpha - \varepsilon_n) \downarrow 0$. Hence, we can now conclude that $\text{STM}_0(\cdot)$ is continuous on $(0, \gamma_2)$. $\square$

Lemma 5.2. For $0 < \alpha \leq \gamma_2$, if

$$\lim_{s \downarrow 0} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_0(s) < \text{TM}_{a,b,0}(\alpha)$$

and

$$\lim_{s \uparrow \alpha} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_0(s) < \text{TM}_{a,b,0}(\alpha),$$

then $\text{TM}_{a,b,0}(\alpha)$ can be attained.

Proof. If that is the case, then by Remark 1.7 we could find some $s \in (0, \alpha)$ such that

$$\frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_0(s) = \text{TM}_{a,b,0}(\alpha).$$

Let u be the optimizer for $\text{STM}_0(s)$, i.e.,

$$\int_{\mathbb{R}^2} H(\nabla u)^2 dx \leq 1, \quad \int_{\mathbb{R}^2} |u(x)|^2 dx = 1$$

and

$$\text{STM}_0(s) = \int_{\mathbb{R}^2} [\exp(s |u(x)|^2) - 1] dx.$$

Then, using the same arguments as in the proof of Remark 1.7, we can verify that

$$v(x) = \left(\frac{s}{\alpha} \right)^{\frac{1}{2}} u(\lambda x)$$

with

$$\lambda = \left[\frac{\frac{s}{\alpha}}{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}} \right]^{\frac{1}{2}} > 0$$

is a maximizer for $\text{TM}_{a,b,0}(\alpha)$. $\square$

Proof of Theorem 1.10 for $\alpha < \gamma_2$. When $\alpha < \gamma_2$, it is easy to see that

$$\lim_{s \uparrow \alpha} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_0(s) = 0 < \text{TM}_{a,b,0}(\alpha).$$

We now show that

$$\lim_{s \downarrow 0} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_0(s) = \alpha.$$

Indeed, let $s_n \downarrow 0$ and H^o -radially symmetric non-increasing functions u_n be such that

$$\int_{\mathbb{R}^2} H(\nabla u_n)^2 dx \leq 1, \quad \int_{\mathbb{R}^2} |u_n(x)|^2 dx = 1$$

and

$$\begin{aligned} \text{STM}_0(s_n) &= \int_{\mathbb{R}^2} [\exp(s_n |u_n(x)|^2) - 1] dx \\ &= s_n + \int_{\mathbb{R}^2} \sum_{j=2}^{\infty} \frac{(s_n |u_n(x)|^2)^j}{j!} dx \\ &\leq s_n + \int_{\mathbb{R}^2} s_n |u_n(x)|^2 [\exp(s_n |u_n(x)|^2) - 1] dx. \end{aligned}$$

By Hölder's inequality, the Sobolev embeddings and the fact that $\text{STM}_0(s) \downarrow 0$ as $s \downarrow 0$, we get

$$\int_{\mathbb{R}^2} |u_n(x)|^2 [\exp(s_n |u_n(x)|^2) - 1] dx \rightarrow 0$$

as $s_n \downarrow 0$. Hence,

$$\lim_{s_n \downarrow 0} \frac{[1 - (\frac{s_n}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s_n}{\alpha}} \text{STM}_0(s_n) = \lim_{s_n \downarrow 0} \frac{\text{STM}_0(s_n)}{\frac{s_n}{\alpha}} = \alpha.$$

Hence, we can now conclude that $\text{TM}_{a,b,0}(\alpha)$ is not achieved only if $\text{TM}_{a,b,0}(\alpha) = \alpha$. We also note here that the mapping

$$\alpha \mapsto \frac{\text{TM}_{a,b,0}(\alpha)}{\alpha}$$

is nondecreasing. As a consequence, there exists some $\alpha_{a,b,0} \geq 0$ such that $\text{TM}_{a,b,0}(\alpha)$ cannot be attained for all $0 < \alpha < \alpha_{a,b,0}$ and $\text{TM}_{a,b,0}(\alpha)$ is achieved for all $\alpha_{a,b,0} < \alpha < \gamma_2$.

Now, if $a > 2$, we will show that $\alpha_{a,b,0} = 0$ by showing that $\text{TM}_{a,b,0}(\alpha) > \alpha$ for all $0 < \alpha < \gamma_2$. Indeed, let us choose a smooth function $v \neq 0$ such that $\int_{\mathbb{R}^2} |v(x)|^2 dx = 1$ and set $v_t(x) = t^{\frac{1}{2}} v(t^{\frac{1}{2}} x)$. Then

$$\begin{aligned} \int_{\mathbb{R}^2} H(\nabla v_t)^2 dx &= t \int_{\mathbb{R}^2} H(\nabla v)^2 dx, \\ \|v_t\|_p &= t^{\frac{p-2}{2p}} \|v\|_p. \end{aligned}$$

Also, for each $t > 0$, we could find some $c_t > 0$ with $u_t = c_t v_t$ such that

$$\left(\int_{\mathbb{R}^2} H(\nabla u_t)^2 dx \right)^{\frac{a}{2}} + \left(\int_{\mathbb{R}^2} |u_t|^2 dx \right)^{\frac{b}{2}} = c_t^a t^{\frac{a}{2}} \left(\int_{\mathbb{R}^2} H(\nabla v)^2 dx \right)^{\frac{a}{2}} + c_t^b = 1.$$

By the implicit function theorem,

$$c'_t = \frac{-\frac{a}{2} c_t^a (\int_{\mathbb{R}^2} H(\nabla v)^2 dx)^{\frac{a}{2}} t^{\frac{a}{2}-1}}{b c_t^{b-1} + a c_t^{a-1} t^{\frac{a}{2}} (\int_{\mathbb{R}^2} H(\nabla v)^2 dx)^{\frac{a}{2}}}.$$

It is also clear that $c_t \uparrow 1$ as $t \downarrow 0$. Then

$$\begin{aligned} \text{TM}_{a,b,0}(\alpha) &\geq \int_{\mathbb{R}^2} [\exp(\alpha |u_t(x)|^2) - 1] dx \\ &\geq \alpha c_t^2 \left(1 + \frac{\alpha}{2} c_t^2 t \|v\|_4^4 \right) \rightarrow \alpha \quad \text{as } t \downarrow 0. \end{aligned}$$

We have that when $a > 2$,

$$\begin{aligned} L'_t &= 2c_t c'_t + \frac{\alpha}{2} c_t^4 \|v\|_4^4 + 2\alpha c_t^3 c'_t t \|v\|_4^4 \\ &= (2c_t + 2\alpha c_t^3 t \|v\|_4^4) \frac{-\frac{\alpha}{2} c_t^a (\int_{\mathbb{R}^2} H(\nabla v)^2 dx)^{\frac{a}{2}} t^{\frac{a}{2}-1}}{b c_t^{b-1} + a c_t^{a-1} t^{\frac{a}{2}} (\int_{\mathbb{R}^2} H(\nabla v)^2 dx)^{\frac{a}{2}}} + \frac{\alpha}{2} c_t^4 \|v\|_4^4 \\ &\rightarrow \frac{\alpha}{2} \|v\|_4^4 > 0 \quad \text{as } t \downarrow 0. \end{aligned}$$

Thus, since $L_t \rightarrow 1$ as $t \downarrow 0$, we get

$$L_t = c_t^2 \left(1 + \frac{\alpha}{2} c_t^2 t \|v\|_4^4 \right) > 1 \quad \text{as } t \gtrsim 0.$$

Hence, in the case $a > 2$, $\text{TM}_{a,b,0}(\alpha) > \alpha$, so $\alpha_{a,b,0} = 0$. That means that $\text{TM}_{a,b,0}(\alpha)$ can be achieved for all $0 < \alpha < \gamma_2$. The same arguments hold (i.e., $\text{TM}_{a,b,0}(\alpha)$ can be attained) when $a = 2$ and

$$\gamma_2 > \alpha > \frac{4}{b} \frac{1}{B_2},$$

where

$$B_2 = \sup \frac{\|v\|_4^4}{(\int_{\mathbb{R}^2} H(\nabla v)^2 dx) \|v\|_2^2}.$$

Another way to deal with this situation is that in this case, it is enough to show that there exists some $s > 0$ such that

$$\text{STM}_0(s) \geq \frac{\frac{s}{\alpha}}{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}} \alpha = \frac{s}{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}.$$

Indeed, it is easy to see that

$$\text{STM}_0(s) \geq s + \frac{s^2}{2} B_2,$$

so we need to show that there is $s > 0$ such that

$$s + \frac{s^2}{2} B_2 \geq \frac{s}{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}.$$

That means

$$B_2 \geq \frac{2}{s} \frac{1 - [1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}.$$

Noting that when $a > 2$,

$$\lim_{s \downarrow 0} \frac{2}{s} \frac{1 - [1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}} = 0,$$

we can conclude that such an s exists. When $a = 2$, we have

$$\lim_{s \downarrow 0} \frac{2}{s} \frac{1 - [1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}} = \frac{4}{b} \frac{1}{\alpha}.$$

Hence, in this case, we are able to choose $s > 0$ if $B_2 > \frac{4}{b} \frac{1}{\alpha}$.

When $a \leq 2$, we will show that $\alpha_{a,b,0} > 0$, i.e., $\text{TM}_{a,b,0}(\alpha)$ cannot be achieved when α is small enough (and hence $\text{TM}_{a,b,0}(\alpha) = \alpha$). To prove this fact, it is enough to show that for $\alpha \gtrsim 0$,

$$\int_{\mathbb{R}^2} [\exp(\alpha |u(x)|^2) - 1] dx < \alpha \quad \text{for all } u : \left(\int_{\mathbb{R}^2} H(\nabla u)^2 dx \right)^{\frac{a}{2}} + \left(\int_{\mathbb{R}^2} |u|^2 dx \right)^{\frac{b}{2}} \leq 1.$$

We first note that by definition, for any $\gamma \in (0, \gamma_2)$,

$$\int_{\mathbb{R}^2} \left[\exp \left(\gamma \left| \frac{u}{\|H(\nabla u)\|_2^2} \right|^2 \right) - 1 \right] dx \leq \text{STM}_0(\gamma) \frac{\|u\|_2^2}{\|H(\nabla u)\|_2^2}.$$

Hence,

$$\begin{aligned}\|u\|_{2+2k}^{2+2k} &\leq \text{STM}_0(\gamma) \frac{(k+1)!}{\gamma^{k+1}} \|H(\nabla u)\|_2^{2k} \|u\|_2^2 \quad (\forall k \geq 0) \\ &\leq \text{STM}_0(\gamma) \frac{(k+1)!}{\gamma^{k+1}} \theta^{2k} (1-\theta^a)^{\frac{2}{b}},\end{aligned}$$

where $\theta = (\int_{\mathbb{R}^2} H(\nabla u)^2 dx)^{\frac{1}{2}} \in (0, 1)$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^2} [\exp(\alpha|u(x)|^2) - 1] dx &= \alpha \|u\|_2^2 + \sum_{k=1}^{\infty} \frac{\alpha^{1+k}}{(1+k)!} \|u\|_{2+2k}^{2+2k} \\ &\leq \alpha (1-\theta^a)^{\frac{2}{b}} + \text{STM}_0(\gamma) \sum_{k=1}^{\infty} \frac{\alpha^{1+k}}{(1+k)!} \frac{(k+1)!}{\gamma^{k+1}} \theta^{2k} (1-\theta^a)^{\frac{2}{b}} \\ &= \alpha (1-\theta^a)^{\frac{2}{b}} + \text{STM}_0(\gamma) \sum_{k=1}^{\infty} \frac{\alpha^{1+k}}{\gamma^{k+1}} \theta^{2k} (1-\theta^a)^{\frac{2}{b}} \\ &= \alpha (1-\theta^a)^{\frac{2}{b}} + \text{STM}_0(\gamma) \frac{\alpha}{\gamma} (1-\theta^a)^{\frac{2}{b}} \sum_{k=1}^{\infty} \frac{\alpha^k}{\gamma^k} \theta^{2k} \\ &= \alpha (1-\theta^a)^{\frac{2}{b}} + \text{STM}_0(\gamma) \frac{\alpha}{\gamma} (1-\theta^a)^{\frac{2}{b}} \left[\frac{1}{1 - \frac{\alpha}{\gamma} \theta^2} - 1 \right] \\ &= \alpha (1-\theta^a)^{\frac{2}{b}} + \text{STM}_0(\gamma) \frac{\alpha}{\gamma} (1-\theta^a)^{\frac{2}{b}} \left[\frac{\alpha \theta^2}{\gamma - \alpha \theta^2} \right].\end{aligned}$$

So it is enough to choose α such that

$$\text{STM}_0(\gamma) \frac{\alpha}{\gamma} (1-\theta^a)^{\frac{2}{b}} \left[\frac{\alpha \theta^2}{\gamma - \alpha \theta^2} \right] < \alpha [1 - (1-\theta^a)^{\frac{2}{b}}],$$

i.e.,

$$\alpha < \frac{\gamma [1 - (1-\theta^a)^{\frac{2}{b}}]}{\text{STM}_0(\gamma) \frac{(1-\theta^a)^{\frac{2}{b}}}{\gamma} \theta^2 + \theta^2 [1 - (1-\theta^a)^{\frac{2}{b}}]}, \quad \forall \theta \in (0, 1).$$

But this is totally possible since by L'Hôpital's rule and the fact that $a \leq 2$,

$$\lim_{\theta \rightarrow 1} \frac{\gamma [1 - (1-\theta^a)^{\frac{2}{b}}]}{\text{STM}_0(\gamma) \frac{(1-\theta^a)^{\frac{2}{b}}}{\gamma} \theta^2 + \theta^2 [1 - (1-\theta^a)^{\frac{2}{b}}]} > 0$$

and

$$\lim_{\theta \rightarrow 0} \frac{\gamma [1 - (1-\theta^a)^{\frac{2}{b}}]}{\text{STM}_0(\gamma) \frac{(1-\theta^a)^{\frac{2}{b}}}{\gamma} \theta^2 + \theta^2 [1 - (1-\theta^a)^{\frac{2}{b}}]} > 0.$$

This completes the proof □

Proof of Theorem 1.10 for $\alpha = \gamma_2$ and $b < 2$. We recall that

$$\sup_{\alpha \in (0, \gamma_2)} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_0(\alpha) = \text{TM}_{a,b,0}(\gamma_2).$$

In particular,

$$\sup_{\alpha \in (0, \gamma_2)} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_0(\alpha) = \text{TM}_{a,2,0}(\gamma_2).$$

Hence, by $b < 2$, we have

$$\lim_{\alpha \uparrow \gamma_2} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_0(\alpha) = \lim_{\alpha \uparrow \gamma_2} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]^{\frac{2}{b}}}{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]}{\frac{\alpha}{\gamma_2}} \text{STM}_0(\alpha) = 0,$$

and by a proof similar to the above one,

$$\lim_{\alpha \downarrow 0} \frac{[1 - (\frac{\alpha}{\gamma_2})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{\alpha}{\gamma_2}} \text{STM}_0(\alpha) = \gamma_2.$$

Hence, we similarly have that $\text{TM}_{a,b,0}(\gamma_2)$ can be achieved if $\text{TM}_{a,b,0}(\gamma_2) > \gamma_2$. For example, if $a > 2$, $\text{TM}_{a,b,0}(\gamma_2)$ can be achieved. $\text{TM}_{a,b,0}(\gamma_2)$ can also be attained if $a = 2$ and

$$\gamma_2 > \frac{4 \left(\int_{\mathbb{R}^2} H(\nabla v)^2 dx \right)}{\|v\|_4^4} \quad \left(\text{for some } v \text{ with } \int_{\mathbb{R}^2} |v(x)|^2 dx = 1 \right)$$

or equivalently,

$$\gamma_2 > b > \frac{4}{\gamma_2} \frac{1}{B_2}.$$

The case $a \leq 2$ is more interesting. In this situation, we will use a fact that when H^o is the standard Euclidean norm $|\cdot|$, $\text{TM}_{2,2,0}(4\pi) > 4\pi$ (see [26, 55]). Then by the convex rearrangement arguments, we can get $\text{TM}_{2,2,0}(\gamma_2) > \gamma_2$. We next show that if $a_n \uparrow 2$ and $b_n \uparrow 2$, then

$$\text{TM}_{a_n, b_n, 0}(\gamma_2) \uparrow \text{TM}_{2,2,0}(\gamma_2).$$

As a consequence, we have $\text{TM}_{a,b,0}(\gamma_2) > \gamma_2$ when $(a, b) \approx (2, 2)$. Hence, we can claim that $\text{TM}_{a,b,0}(\gamma_2)$ can be achieved when $(a, b) \approx (2, 2)$. Indeed, for any $\varepsilon > 0$, let u be such that

$$\int_{\mathbb{R}^2} H(\nabla u)^2 dx + \int_{\mathbb{R}^2} |u|^2 dx = 1$$

and

$$\text{TM}_{2,2,0}(\gamma_2) < \int_{\mathbb{R}^2} [\exp(\gamma_2 |u(x)|^2) - 1] dx + \varepsilon.$$

Since $a_n \uparrow 2$ and $b_n \uparrow 2$, we could find some $c_n \uparrow 1$ such that

$$c_n^a \left(\int_{\mathbb{R}^2} H(\nabla u)^2 dx \right)^{\frac{a}{2}} + c_n^b \left(\int_{\mathbb{R}^2} |u|^2 dx \right)^{\frac{b}{2}} = 1.$$

Then

$$\begin{aligned} 0 &\leq \text{TM}_{2,2,0}(\gamma_2) - \text{TM}_{a_n, b_n, 0}(\gamma_2) \\ &\leq \int_{\mathbb{R}^2} [\exp(\gamma_2 |u(x)|^2) - 1] dx - \int_{\mathbb{R}^2} [\exp(\gamma_2 c_n^2 |u(x)|^2) - 1] dx + \varepsilon \\ &= \int_{\mathbb{R}^2} [\exp(\gamma_2 |u(x)|^2) - \exp(\gamma_2 c_n^2 |u(x)|^2)] dx + \varepsilon. \end{aligned}$$

Since $\exp(\gamma_2 |u(x)|^2) - \exp(\gamma_2 c_n^2 |u(x)|^2) \rightarrow 0$ a.e.,

$$0 \leq \exp(\gamma_2 |u(x)|^2) - \exp(\gamma_2 c_n^2 |u(x)|^2) \leq \exp(\gamma_2 |u(x)|^2) - 1 \in L^1$$

and ε is arbitrary, we can conclude that $\text{TM}_{2,2,0}(\gamma_2) - \text{TM}_{a_n, b_n, 0}(\gamma_2) \downarrow 0$. □

Remark 5.3.

$$\lim_{s \downarrow 0} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_0(s)$$

is actually the vanishing phenomenon of $\text{TM}_{a,b,0}(\alpha)$ in the sense of [19, 26]. Similarly,

$$\lim_{s \uparrow \alpha} \frac{[1 - (\frac{s}{\alpha})^{\frac{a}{2}}]^{\frac{2}{b}}}{\frac{s}{\alpha}} \text{STM}_0(s)$$

is the concentration alternative for $\text{TM}_{a,b,0}(\alpha)$. Hence, it can be seen from the above proofs that in the subcritical cases (i.e., $\alpha < \gamma_2$ or $\alpha = \gamma_2$ and $b = N$), the concentration situation cannot occur.

6 Affine Trudinger-Moser type inequalities

In this section, we will apply our results for suitable Finsler norms to get the affine versions of the Trudinger-Moser type inequalities. Thus, we provide a different approach to the results in [13].

6.1 Brunn-Minkowski theory

We recall here some background material from the Brunn-Minkowski theory of convex bodies. The interested reader can refer to [46, 47, 58] and the references therein.

A convex body $K \subset \mathbb{R}^N$ is a convex compact subset of $\mathbb{R}^N$ with nonempty interior. Its support function is defined as $h_K(x) = \sup_{y \in K} \langle x, y \rangle$. We also define the gauge $\|\cdot\|_K$ and radial $r_K(\cdot)$ functions as

$$\begin{aligned}\|x\|_K &= \inf\{\lambda > 0 : x \in \lambda K\}, \\ r_K(x) &= \sup\{\lambda > 0 : \lambda x \in K\}.\end{aligned}$$

It is obvious that $\|x\|_K = \frac{1}{r_K(x)}$.

For $K \subset \mathbb{R}^N$, we also define its polar body by

$$K^\circ = \{x \in \mathbb{R}^N : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

Hence, $h_K = r_{K^\circ}$. We also have

$$|K| = \frac{1}{N} \int_{S^{N-1}} r_K^N(x) dx = \frac{1}{N} \int_{S^{N-1}} \|x\|_K^{-N} dx.$$

For each $\lambda \in [0, 1]$, the general L^p centroid body of K is the convex body

$$\Gamma_{\lambda,p} K = (1-\lambda)\Gamma_p^+ K + \lambda\Gamma_p^- K,$$

where $\Gamma_p^- K = \Gamma_p^+(-K)$. Here, $\Gamma_p^+ K$ is the asymmetric L^p centroid body of K that is defined by

$$\begin{aligned}h_{\Gamma_p^+ K}^p(x) &= \frac{1}{\alpha_{N,p}|K|} \int_K \langle x, y \rangle_+^p dy, \\ \alpha_{N,p} &= \frac{\omega_{N+p-2}}{(N+p)\omega_N\omega_{p-1}}.\end{aligned}$$

Hence, we get

$$h_{\Gamma_{\lambda,p} K}^p(x) = \frac{1}{\alpha_{N,p}|K|} \int_K [(1-\lambda)\langle x, y \rangle_+^p + \lambda\langle x, y \rangle_-^p] dy.$$

Here, $\langle x, y \rangle_+ = \max\{\langle x, y \rangle, 0\}$ and $\langle x, y \rangle_- = \max\{-\langle x, y \rangle, 0\}$.

We mention here the L^p Busemann-Petty centroid inequality $|\Gamma_{\lambda,p} K| \geq |K|$. The equality occurs if and only if K is an origin-centered ellipsoid. This affine isoperimetric inequality was first investigated in [48] to generalize the classical Busemann-Petty centroid inequality by Petty [53].

Now, we define the general L^p affine energy for $f \in W^{1,p}(\mathbb{R}^N)$ by

$$\begin{aligned}\mathcal{E}_{\lambda,p}(f) &= c_{N,p} \left(\int_{S^{N-1}} \left(\int_{\mathbb{R}^N} [(1-\lambda)(\nabla f(x) \cdot \sigma)_+^p + \lambda(\nabla f(x) \cdot \sigma)_-^p] dx \right)^{-\frac{N}{p}} d\sigma \right)^{-\frac{1}{N}}, \\ c_{N,p} &= (N\omega_N)^{\frac{1}{N}} \left(\frac{N\omega_N\omega_{p-1}}{\omega_{N+p-2}} \right)^{\frac{1}{p}}.\end{aligned}$$

Here,

$$0 \leq \lambda \leq 1, \quad (\nabla f(x) \cdot \sigma)_+ = \max\{\nabla f(x) \cdot \sigma, 0\} \quad \text{and} \quad (\nabla f(x) \cdot \sigma)_- = \max\{-\nabla f(x) \cdot \sigma, 0\}.$$

It can be noted that when $\lambda = \frac{1}{2}$, $\mathcal{E}_{\frac{1}{2},p}(f)$ is the L^p affine energy $\mathcal{E}_p(f)$ introduced in [13] while $\mathcal{E}_{0,p}(f)$ is exactly the asymmetric L^p affine energy $\mathcal{E}_p^+(f)$ studied in [24].

6.2 Affine Trudinger-Moser type inequalities from Finsler Trudinger-Moser type inequalities

Now, fix $f \in C_0^\infty(\mathbb{R}^2)$. We will use the following Finsler norms:

$$H_f(x) = \left[\int_{S^1} \|y\|_{2,\lambda,f}^{-4} [(1-\lambda)\langle x, y \rangle_+^2 + \lambda\langle x, y \rangle_-^2] dy \right]^{\frac{1}{2}},$$

$$\|x\|_{p,\lambda,f} = \left[\int_{\mathbb{R}^2} [(1-\lambda)\langle x, \nabla f(y) \rangle_+^2 + \lambda\langle x, \nabla f(y) \rangle_-^2] dy \right]^{\frac{1}{2}}.$$

We also set

$$K_{\lambda,2}(f) = \{x : H_f^o(x) \leq 1\},$$

$$L_{\lambda,2}(f) = \{x : \|x\|_{2,\lambda,f} \leq 1\}.$$

We note that

$$\begin{aligned} \int_{\mathbb{R}^2} H(\nabla f(x))^2 dx &= \int_{\mathbb{R}^2} \int_{S^1} \|y\|_{2,\lambda,f}^{-4} [(1-\lambda)\langle \nabla f(x), y \rangle_+^2 + \lambda\langle \nabla f(x), y \rangle_-^2] dy dx \\ &= \int_{S^1} \|y\|_{2,\lambda,f}^{-4} \int_{\mathbb{R}^2} [(1-\lambda)\langle \nabla f(x), y \rangle_+^2 + \lambda\langle \nabla f(x), y \rangle_-^2] dx dy \\ &= \int_{S^1} \|y\|_{2,\lambda,f}^{-2} dy \\ &= 2|L_{\lambda,2}(f)| = \left(\frac{\mathcal{E}_{\lambda,2}(f)}{c_{2,2}} \right)^{-2}. \end{aligned}$$

Moreover, from

$$\begin{aligned} h_{K_{\lambda,2}(f)}^2(x) &= \int_{S^1} \|y\|_{2,\lambda,f}^{-4} [(1-\lambda)\langle x, y \rangle_+^2 + \lambda\langle x, y \rangle_-^2] dy \\ &= 4 \int_{S^1} \int_0^{\|y\|_{2,\lambda,f}^{-1}} [(1-\lambda)\langle x, y \rangle_+^2 + \lambda\langle x, y \rangle_-^2] r^3 dr dy \\ &= 4 \int_{L_{\lambda,2}(f)} [(1-\lambda)\langle x, y \rangle_+^2 + \lambda\langle x, y \rangle_-^2] dy \\ &= 4\alpha_{2,2}|L_{\lambda,2}(f)|h_{\Gamma_{\lambda,2}L_{\lambda,2}(f)}^2(x), \end{aligned}$$

we can deduce that

$$K_{\lambda,2}(f) = [4\alpha_{2,2}|L_{\lambda,2}(f)|]^{\frac{1}{2}} \Gamma_{\lambda,2} L_{\lambda,2}(f).$$

Thus, by the L^2 Busemann-Petty centroid inequality $|\Gamma_{\lambda,2} L_{\lambda,2}(f)| \geq |L_{\lambda,2}(f)|$, we obtain

$$|K_{\lambda,2}(f)| \geq 4\alpha_{2,2} \left(\frac{1}{2} \left(\frac{\mathcal{E}_{\lambda,2}(f)}{c_{2,2}} \right)^{-2} \right)^2. \quad (6.1)$$

The equality occurs when $L_{\lambda,2}(f)$ is an origin-centered ellipsoid.

Hence,

$$\kappa_2 = |K_{\lambda,2}(f)| \geq 4\alpha_{2,2} \left(\frac{1}{2} \left(\frac{\mathcal{E}_{\lambda,2}(f)}{c_{2,2}} \right)^{-2} \right)^2$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} H(\nabla f(x))^2 dx &= \left(\frac{\mathcal{E}_{\lambda,2}(f)}{c_{2,2}} \right)^{-2} \\ &= 4\alpha_{2,2} \left(\frac{1}{2} \left(\frac{\mathcal{E}_{\lambda,2}(f)}{c_{2,2}} \right)^{-2} \right)^2 \frac{1}{\alpha_{2,2}} \left(\frac{\mathcal{E}_{\lambda,2}(f)}{c_{2,2}} \right)^2 \end{aligned}$$

$$\leq \kappa_2 \frac{\mathcal{E}_{\lambda,2}^2(f)}{\omega_2}. \quad (6.2)$$

Hence, by Theorem 1.2, we obtain for $0 \leq \alpha < 4\omega_2$,

$$\sup_{f \in C_0^\infty(\mathbb{R}^2), \mathcal{E}_{\lambda,2}^2(f) \leq 1} \frac{1}{\int_{\mathbb{R}^2} |f(x)|^2 dx} \int_{\mathbb{R}^2} [\exp(\alpha |f(x)|^2) - 1] dx < \infty.$$

We also note that with a mild modification in the proof of Theorem 1.6, we get that for $\tau > 0$, $a > 0$ and $0 < b \leq 2$,

$$\sup_{(\int_{\mathbb{R}^2} H(\nabla u)^2 dx)^{\frac{a}{2}} + \tau (\int_{\mathbb{R}^2} |u(x)|^2 dx)^{\frac{b}{2}} \leq 1} \int_{\mathbb{R}^2} [\exp(\gamma_2 |u(x)|^2) - 1] dx < \infty.$$

Hence, using (6.2) and a simple argument, we can deduce that

$$\sup_{(\mathcal{E}_{\lambda,2}^2(f))^{\frac{a}{2}} + (\int_{\mathbb{R}^2} |f(x)|^2 dx)^{\frac{b}{2}} \leq 1} \int_{\mathbb{R}^2} [\exp(4\omega_2 |f(x)|^2) - 1] dx < \infty.$$

References

- Adachi S, Tanaka K. Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents. *Proc Amer Math Soc*, 1999, 128: 2051–2057
- Adimurthi, Sandeep K. A singular Trudinger-Moser embedding and its applications. *NoDEA Nonlinear Differential Equations Appl*, 2007, 13: 585–603
- Adimurthi, Yang Y. An interpolation of Hardy inequality and Trudinger-Moser inequality in $\mathbb{R}^N$ and its applications. *Int Math Res Not IMRN*, 2010, 13: 2394–2426
- Alvino A, Ferone V, Mercaldo A, et al. Finsler Hardy-Kato's inequality. *J Math Anal Appl*, 2019, 470: 360–374
- Alvino A, Ferone V, Trombetti G, et al. Convex symmetrization and applications. *Ann Inst H Poincaré Anal Non Linéaire*, 1997, 14: 275–293
- Caffarelli L, Kohn R, Nirenberg L. First order interpolation inequalities with weights. *Compos Math*, 1984, 53: 259–275
- Çağil A. Finsler geometry and its applications to electromagnetism. MSc Thesis. Ankara: Middle East Technical University, 2003
- Carleson L, Chang S-Y A. On the existence of an extremal function for an inequality of J. Moser. *Bull Sci Math*, 1986, 110: 113–127
- Cassani D, Sani F, Tarsi C. Equivalent Moser type inequalities in $\mathbb{R}^2$ and the zero mass case. *J Funct Anal*, 2014, 267: 4236–4263
- Chang S-Y A, Yang P C. The inequality of Moser and Trudinger and applications to conformal geometry. *Comm Pure Appl Math*, 2003, 56: 1135–1150
- Chen L, Lu G, Zhang C. Sharp weighted Trudinger-Moser-Adams inequalities on the whole space and the existence of their extremals. *Calc Var Partial Differential Equations*, 2019, 58: 132
- Chen L, Lu G, Zhu M. Existence and nonexistence of extremals for critical Adams inequalities in $\mathbb{R}^4$ and Trudinger-Moser inequalities in $\mathbb{R}^2$. *Adv Math*, 2020, 368: 107143
- Cianchi A, Lutwak E, Yang D, et al. Affine Moser-Trudinger and Morrey-Sobolev inequalities. *Calc Var Partial Differential Equations*, 2009, 36: 419–436
- Clayton J D. On Finsler geometry and applications in mechanics: Review and new perspectives. *Adv Math Phys*, 2015, 2015: 828475
- Csató G, Roy P. Extremal functions for the singular Moser-Trudinger inequality in 2 dimensions. *Calc Var Partial Differential Equations*, 2015, 54: 2341–2366
- De Figueiredo D G, do Ó J M, Ruf B. Elliptic equations and systems with critical Trudinger-Moser nonlinearities. *Discrete Contin Dyn Syst*, 2011, 30: 455–476
- Della Pietra F, di Blasio G, Gavitone N. Anisotropic Hardy inequalities. *Proc Roy Soc Edinburgh Sect A*, 2018, 148: 483–498
- do Ó J M. N -Laplacian equations in $\mathbb{R}^N$ with critical growth. *Abstr Appl Anal*, 1997, 2: 301–315
- do Ó J M, Sani F, Tarsi C. Vanishing-concentration-compactness alternative for the Trudinger-Moser inequality in $\mathbb{R}^N$. *Commun Contemp Math*, 2018, 20: 1650036
- Dong M, Lam N, Lu G. Singular Trudinger-Moser inequalities, Caffarelli-Kohn-Nirenberg inequalities and their extremal functions. *Nonlinear Anal*, 2018, 173: 75–98

- 21 Dong M, Lu G. Best constants and existence of maximizers for weighted Moser-Trudinger inequalities. *Calc Var Partial Differential Equations*, 2016, 55: 55–88
- 22 Duy N T, Lam N, Phi L L. Sharp affine Trudinger-Moser inequalities: A new argument. *Canad Math Bull*, 2021, 64: 765–778
- 23 Flucher M. Extremal functions for the Trudinger-Moser inequality in 2 dimensions. *Comment Math Helv*, 1992, 67: 471–497
- 24 Haberl C, Schuster F E, Xiao J. An asymmetric affine Pólya-Szegő principle. *Math Ann*, 2012, 352: 517–542
- 25 Ibrahim S, Masmoudi N, Nakanishi K. Trudinger-Moser inequality on the whole plane with the exact growth condition. *J Eur Math Soc JEMS*, 2015, 17: 819–835
- 26 Ishiwata M. Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in $\mathbb{R}^N$. *Math Ann*, 2011, 351: 781–804
- 27 Ishiwata M, Nakamura M, Wadade H. On the sharp constant for the weighted Trudinger-Moser type inequality of the scaling invariant form. *Ann Inst H Poincaré Anal Non Linéaire*, 2014, 31: 297–314
- 28 Lam N. Equivalence of sharp Trudinger-Moser-Adams inequalities. *Commun Pure Appl Anal*, 2017, 16: 973–997
- 29 Lam N. Maximizers for the singular Trudinger-Moser inequalities in the subcritical cases. *Proc Amer Math Soc*, 2017, 145: 4885–4892
- 30 Lam N. Sharp subcritical and critical Trudinger-Moser inequalities on $\mathbb{R}^2$ and their extremal functions. *Potential Anal*, 2017, 46: 75–103
- 31 Lam N. Sharp Trudinger-Moser inequalities with monomial weights. *NoDEA Nonlinear Differential Equations Appl*, 2017, 24: 39
- 32 Lam N. Optimizers for the singular Trudinger-Moser inequalities in the critical case in $\mathbb{R}^2$. *Math Nachr*, 2018, 291: 2272–2287
- 33 Lam N. General sharp weighted Caffarelli-Kohn-Nirenberg inequalities. *Proc Roy Soc Edinburgh Sect A*, 2019, 149: 691–718
- 34 Lam N, Lu G. Sharp Moser-Trudinger inequality on the Heisenberg group at the critical case and applications. *Adv Math*, 2012, 231: 3259–3287
- 35 Lam N, Lu G. A new approach to sharp Moser-Trudinger and Adams type inequalities: A rearrangement-free argument. *J Differential Equations*, 2013, 255: 298–325
- 36 Lam N, Lu G. Sharp affine and improved Moser-Trudinger-Adams type inequalities on unbounded domains in the spirit of Lions. *J Geom Anal*, 2017, 27: 300–334
- 37 Lam N, Lu G. Sharp constants and optimizers for a class of Caffarelli-Kohn-Nirenberg inequalities. *Adv Nonlinear Stud*, 2017, 17: 457–480
- 38 Lam N, Lu G, Zhang L. Equivalence of critical and subcritical sharp Trudinger-Moser-Adams inequalities. *Rev Mat Iberoam*, 2017, 33: 1219–1246
- 39 Lam N, Lu G, Zhang L. Existence and nonexistence of extremal functions for sharp Trudinger-Moser inequalities. *Adv Math*, 2019, 352: 1253–1298
- 40 Lam N, Lu G, Zhang L. Sharp singular Trudinger-Moser inequalities under different norms. *Adv Nonlinear Stud*, 2019, 19: 239–261
- 41 Li Y X. Moser-Trudinger inequality on compact Riemannian manifolds of dimension two. *J Partial Differ Equ*, 2001, 14: 163–192
- 42 Li Y X. Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds. *Sci China Ser A*, 2005, 48: 618–648
- 43 Li Y X, Ruf B. A sharp Moser-Trudinger type inequality for unbounded domains in $\mathbb{R}^n$. *Indiana Univ Math J*, 2008, 57: 451–480
- 44 Lin K. Extremal functions for Moser’s inequality. *Trans Amer Math Soc*, 1996, 348: 2663–2671
- 45 Lu G, Tang H. Best constants for Moser-Trudinger inequalities on high dimensional hyperbolic spaces. *Adv Nonlinear Stud*, 2013, 13: 1035–1052
- 46 Lutwak E. The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem. *J Differential Geom*, 1993, 38: 131–150
- 47 Lutwak E. The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas. *Adv Math*, 1996, 118: 244–294
- 48 Lutwak E, Yang D, Zhang G. L_p affine isoperimetric inequalities. *J Differential Geom*, 2000, 56: 111–132
- 49 Masmoudi N, Sani F. Trudinger-Moser inequalities with the exact growth condition in $\mathbb{R}^N$ and applications. *Comm Partial Differential Equations*, 2015, 40: 1408–1440
- 50 Mercaldo A, Sano M, Takahashi F. Finsler Hardy inequalities. *Math Nachr*, 2020, 293: 2370–2398
- 51 Moser J. A sharp form of an inequality by N. Trudinger. *Indiana Univ Math J*, 1971, 20: 1077–1092
- 52 Nguyen V H, Takahashi F. On a weighted Trudinger-Moser type inequality on the whole space and related maximizing problem. *Differential Integral Equations*, 2018, 31: 785–806
- 53 Petty C M. Centroid surfaces. *Pacific J Math*, 1961, 11: 1535–1547

- 54 Pohožaev S I. On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Dokl Akad Nauk SSSR, 1965, 165: 36–39
- 55 Ruf B. A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$. J Funct Anal, 2005, 219: 340–367
- 56 Russell B, Stepney S. Applications of Finsler geometry to speed limits to quantum information processing. Internat J Found Comput Sci, 2014, 25: 489–505
- 57 Ruzhansky M, Suragan D. Anisotropic L^2 -weighted Hardy and L^2 -Caffarelli-Kohn-Nirenberg inequalities. Commun Contemp Math, 2017, 19: 1750014
- 58 Schneider R. Convex Bodies: The Brunn-Minkowski Theory, 2nd ed. Encyclopedia of Mathematics and Its Applications, vol. 151. Cambridge: Cambridge University Press, 2014
- 59 Shen Z. Riemann-Finsler geometry with applications to information geometry. Chin Ann Math Ser B, 2006, 27: 73–94
- 60 Trudinger N S. On imbeddings into Orlicz spaces and some applications. J Math Mech, 1967, 17: 473–483
- 61 Wang G, Xia C. Blow-up analysis of a Finsler-Liouville equation in two dimensions. J Differential Equations, 2012, 252: 1668–1700
- 62 Wulff G. Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflächen. Z Kristallogr Cryst Mater, 1901, 34: 449–530
- 63 Yudovich V I. Some estimates connected with integral operators and with solutions of elliptic equations. Dokl Akad Nauk SSSR, 1961, 138: 805–808
- 64 Zhou C, Zhou C. Extremal functions of Moser-Trudinger inequality involving Finsler-Laplacian. Commun Pure Appl Anal, 2018, 17: 2309–2328
- 65 Zhou C, Zhou C. Moser-Trudinger inequality involving the anisotropic Dirichlet norm $(\int_{\Omega} F^N(\nabla u) dx)^{\frac{1}{N}}$ on $W_0^{1,N}(\Omega)$. J Funct Anal, 2019, 276: 2901–2935